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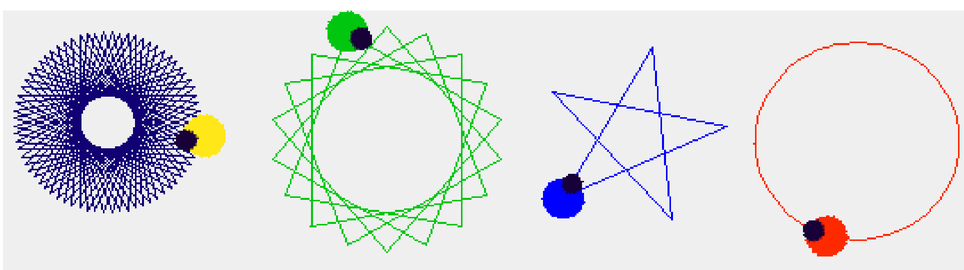
Squeak: **Geometry of Forward and Turn**

Sometimes they are edgy, sometimes they are smooth, sometimes they look like circles, yet sometimes they are so square. Are they Moody? Not at all, they are very predictable. With geometry you can prove it. Are they stubborn? No at all, with geometry you can even convince them to follow your will.

Investigation of the script: Forward by A; Turn by B

Look at the different patterns that are drawn by an object that follows the script: **Object forward by A; Object turn by B**; (with "pen down") as you change the values of A and B.
Try: A=5, B=5; A=100, B=100; A=100, B=85; A=100, B=86; and others of your choice.

All the patterns below were created by scripts of the form:
Forward by A; Turn by B;



Questions to ponder:

1. When the script is: forward by 10, turn by 10, you see a circle. Is the drawing really a circle? What are we drawing with scripts of this type?
2. What can you say, and prove, about the circles you get?
3. Sometimes (check: forward by 100; turn by 80) we get more than one polygon. When does this happen and why?
4. Can we control the number of vertices we get? If your answer is yes, explain **how**. If your answer is no, explain **why**.

Teachers guide for:

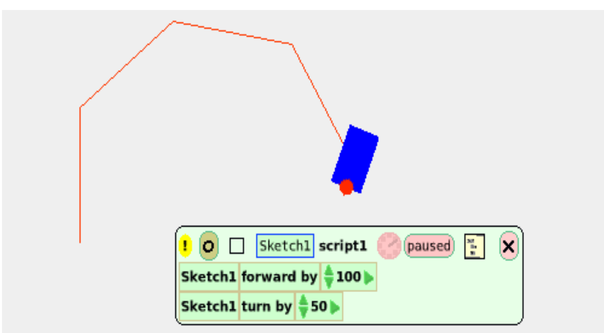
Squeak: **Geometry project***

An Investigation of the script: Forward by A; Turn by B;
Aligned with NCTM and IL State Standards. See reference below.

Goals: Use what we know from geometry to make sense of, and gain insight into Squeak's beautiful graphics. In particular: Use congruent triangles and the definition of a circle to:

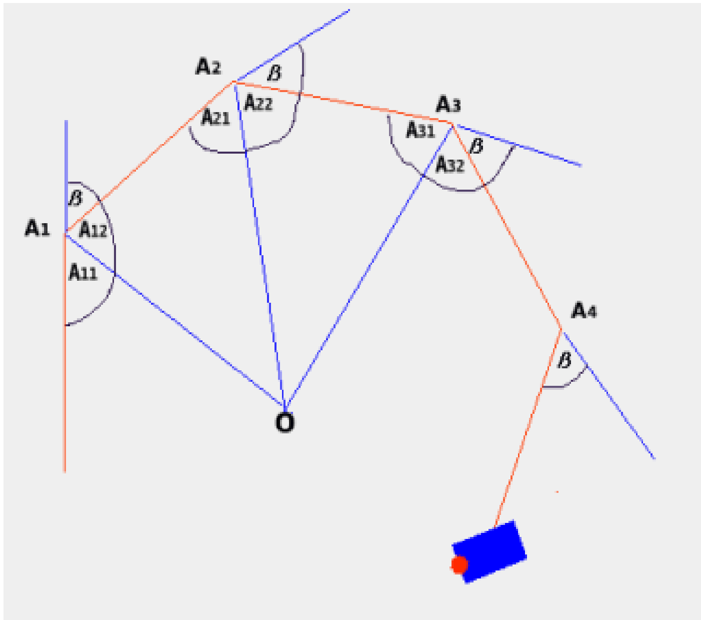
- **Find, define and prove** that the result of every traced script of the form: **forward by A** (distance)**; turn by B**; (an angle-measure in degrees) is **a sequence of equal length segments that are circumscribed by an outer circle and that inscribe an inner circle.** In other words: all the segments are between 2 circles, an outer circle that contains all the vertices of the segments, and an inner one that contains all the midpoints of the segments.
- **Each segment subtends an arc of B degrees on the circumscribed circle.** For example: the script: forward by A; turn by 90; when executed with "pen down" yields a square with sides of length A, the same script with B=60 yields an hexagon with sides of length A..

Analysis:
The very basic element of the script forward by A; turn by B is a segment of length A. When the script is executed many times, we get many segments of length A, each is making an angle (180-B) degrees with its predecessor. See Picture1.



Picture 1

Theorem 1: There is a point O from which all the segments vertices are distance R. In other words: all the vertices are on one circle, its center at O and its radius is R.



Picture 2

Outline of proof:

Let A_1A_2, A_2A_3, A_3A_4…A_nA_{n+1} be segments drawn by an object with script: forward by A, turn by B; See picture 2.

Let O be the intersection of the bisector of angle A_1 and the bisector of angle A_2.

We are going to prove that OA_1=OA_2=OA_3=OA_4=…=OA_n=R.

Triangle A_1A_2O is isosceles by the bisected angles, therefore: A_1O=A_2O=R

Triangles A_1A_2O and A_3A_2O are congruent (S.A.S)

Therefore:A_1O=A_3O =R

Also: <)A_{21} = <)A_{31} (CPCTC) so <)A_{31}=1/2<)A_3 therefore OA_3 bisects angle A_3.

We have two consecutive angle bisectors, so we can repeat the same arguments, (replacing A_1, A_2 by A_2, A_3 respectively and A_3 by A_4) and prove that OA_4 =OA_2=R and so on so forth for every edge of the segments.

Theorem 2: The central angle opposite each segment is of size B.

Proof: In isosceles triangle A_1A2O (See theorem 1)

m<)A_{12}+m<)A_{21}+m<)A_1OA_2=180 (triangle) and m<)A_{21}+m<)A_{22}+m<)B=180 (straight line)

<)A_{12}=<)A_{21}=<)A_{22} therefore m<)A_1OA_2=180- [m<)A_{21}+m<)A_{22}]= β.

Since all the triangles OA_iA_{i+1} are congruent, each of the inner circle angles equal B.

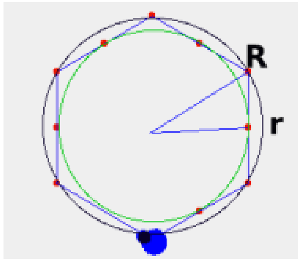
Theorem 3: All the midpoints of the original segments lie on one circle with center at O and the segments inscribe this circle.

Outline of proof:

Let M_i , i=1,2,.. n be the midsection points.

Look at the segments M_iO . They all are of the same length (all triangles A_iM_iO are congruent).

Since each triangle is isosceles, in each triangle, M_iO is also an altitude, perpendicular to the original segment A_iA_{i+1}.
Therefore: all the points M_i are on a circle that O is its center and each original segment is tangential to this circle at M_i, another way to say that the circle is inscribed by the segments. See picture 3.

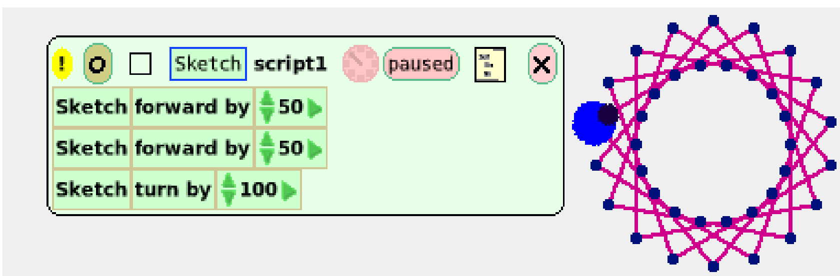


Picture 3

In Conclusion:
All the segments drawn by the script forward by A; turn by B; are between those 2 circles. Their common center O is at the intersection of the angle bisectors.
The radius of the inner circle is the distance to the midpoint of each original segment.
The radius of the outer circle is the distance to the vertex of each original segment.

A nice way to demonstrate that the outer circle contains the vertices and the inner one contains the midpoints is by executing the following script 1 twice; First choose pen trail style to be "line" and run the script until you see no changes happening. Stop the script. For second run, choose a new pen color and change the pen trail style to "dots". Start the script again and the circles will form. See picture 4.

Script 1: forward by A/2, forward by A/2, turn by B.



Picture 4

Clues to the Ponder Questions

1) Try increasing the size of the turn angle. The larger the angle the more the polygon reveals itself.
2) Done in proof above
3) You get one regular polygon when there is an integer N so that N*B=360 (when we limit the angles to integers, we can sum it: the angle B is a factor of 360).

4) For N vertices polygon make the turn (360/N). One can get more complicated shapes by using an expression like: (M*360/N) for the measure of the turn, where M and N are whole numbers.

Reference:

In line with the following Geometry standards by NCTM for grades 9-12:

- Analyze characteristics and properties of two and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
- 1. Use visualization, spatial reasoning and geometric modeling to solve problems. (Use geometric ideas to solve problems in, and gain insight into, other disciplines and other areas of interest such as art and architecture)

In line with Illinois Mathematics Standards; 9.c.4a, 9.c.4b, 9.c.4c and 9.a.5, 9.c.5a, 9.c.5b

***All the SQUEAK you need can be cover in half an hour!!!**

Key Words:

Squeak
Polygons
Congruent triangles
Central Angle